Generalized quantization condition in topological insulator

Yuanpei Lan[1], Shaolong Wan[1],[*] and Shou-Cheng Zhang[2]

[1] *Department of Modern Physics, University of Science and Technology of China, Hefei, 230026,* **P. R. China**
[2] *Department of Physics, Stanford University, Stanford, CA94305-4045,* **USA**

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The topological magnetoelectric effect (TME) is the fundamental quantization effect for topological insulators in units of the fine structure constant α. In [Phys. Rev. Lett. 105, 166803(2010)], a topological quantization condition of the TME is given under orthogonal incidence of the optical beam, in which the wave length of the light or the thickness of the TI film must be tuned to some commensurate values. This fine tuning is difficult to realize experimentally. In this article, we give manifestly $SL(2,\mathbb{Z})$ covariant expressions for Kerr and Faraday angles at oblique incidence at a topological insulator thick film. We obtain a generalized quantization condition independent of material details, and propose a more easily realizable optical experiment, in which only the incidence angle is tuned, to directly measure the topological quantization associated with the TME.

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1. INTRODUCTION

Recently, the time-reversal invariant topological insulator (TI) has been investigated extensively [1–4]. The concept of the topological insulator can be defined both within the topological field theory(TFT)[5] and topological band theory(TBT)[6–8]. The TFT is generally valid for interacting systems and describing a quantized magnetoelectric response, i.e. topological magnetoelectric effect(TME) [5]. Moveover, the TFT reduces exactly to the TBT in the non-interacting limit [9]. TME in topological insulator is defined as a magnetization induced by an electric field or a charge polarization induced by a magnetic field, this effect is essentially a surface effect although it looks like a bulk response. The most important feature of TME is the quantization of the magnetoelectric coefficient $\partial M/\partial E$ or $\partial P/\partial B$ in odd units of the fine structure constant [5, 10] However, since the standard Maxwell term has the same scaling dimension as the topological term, non-universal materials constants such as ε and μ can mask the exact quantization of the TME[5, 11]. Hence, in order to observe the topological quantization of the TME, one should design an experiment to remove the dependence of the non-topological material properties of the TI such as ε and μ. Recently new proposals have been made to remove the dependence on those non-topological material constants [11, 12]. In Ref.[11], one considers a thick film of topological insulator with two surfaces, with vacuum on one side and a substrate on the other. With light rays normally incident at the slab, the combination of Kerr and Faraday angles measured at reflectivity minima or maxima can directly give topological quantization of the TME. In a more special case, Ref.[12] considers a topological insulator thin film weakly exchange coupled to ferromagnet and finds that in the low frequency limit, both the Faraday rotation and the Kerr rotation are universal, dependent only on the vacuum fine structure constant. But these quantization condition apply only to orthogonal incidence, and the thickness of TI film or the frequency of the incident light must be tuned to specific values which may be difficult in practical experiments.

In this paper, we consider the general case of oblique incidence, and obtain a generalized topological quantization condition, in which the reflectivity minima can be easily realized by changing incidence angle rather than by tuning the film thickness. This article is organized as follows. In Sec. II, we present Kerr and Faraday rotation at a unique interface with oblique incidence. In Sec. III, Kerr and Faraday rotation on a topological insulator thick film at oblique incidence are given. In Sec. IV, Generalized quantization condition in TME is obtained, which is easily realizable optical experiments. The conclusion is given in Sec. V.

[*] Corresponding author; Electronic address: slwan@ustc.edu.cn

2. KERR AND FARADAY ROTATION AT A UNIQUE INTERFACE WITH OBLIQUE INCIDENCE

According to TFT, the effective Lagrangian of a topological insulator is given by [1, 5]

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\theta = \frac{1}{8\pi}(\varepsilon\vec{E}^2 - \frac{1}{\mu}\vec{B}^2) + \frac{\theta}{2\pi}\frac{\alpha}{2\pi}\vec{E}\cdot\vec{B} \tag{1}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, ε and μ are the dielectric constant and magnetic permeability, respectively, θ is the axion angle [13], and α is the fine structure constant. Under the time-reversal transformation, $\vec{E} \longrightarrow \vec{E}$, $\vec{B} \longrightarrow -\vec{B}$, so for a periodic system, there are only two values of θ, namely $\theta = 0$ and $\theta = \pi$ (modulo 2π), that give a time-reversal symmetric theory.

In the framework of $SL(2,\mathbb{Z})$ electric-magnetic duality symmetry, the constitutive relation can be written in the compact form [14]

$$\begin{pmatrix} \vec{D} \\ 2\alpha\vec{B} \end{pmatrix} = \mathcal{M}\begin{pmatrix} 2\alpha\vec{E} \\ \vec{H} \end{pmatrix} \tag{2}$$

with

$$\mathcal{M} = \frac{1}{c}\frac{2\alpha}{c\varepsilon}\begin{pmatrix} \frac{\theta^2}{4\pi^2} + (\frac{c\varepsilon}{2\alpha})^2 & \frac{\theta}{2\pi} \\ \frac{\theta}{2\pi} & 1 \end{pmatrix} \tag{3}$$

where $\det(\mathcal{M}) = \frac{1}{c^2}$ is duality invariant under a general $SL(2,\mathbb{Z})$ transformation. On the other hand, in the presence of a non-trivial θ angle Maxwell's equations still allow propagating wave solutions with $\omega = ck$, all fields orthogonal to the direction of propagation, and

$$\begin{pmatrix} 2\alpha\vec{E} \\ \vec{H} \end{pmatrix} = c\hat{k}\times\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \vec{D} \\ 2\alpha\vec{B} \end{pmatrix} \tag{4}$$

where $\hat{k} = \frac{\vec{k}}{|\vec{k}|}$ is the unit wave vector. That is, $\vec{E}\perp\vec{B}$ and $\vec{D}\perp\vec{H}$. However the 2-bein defined by $\hat{D}$, $\hat{H}$ is no longer aligned with the $\hat{E}$, $\hat{B}$ 2-bein due to the topological θ term, which is distinguished from the classic electrodynamics [15].

For simplicity, we denote

$$\vec{\mathcal{E}} = \begin{pmatrix} 2\alpha\vec{E} \\ \vec{H} \end{pmatrix}, \qquad \vec{\mathcal{D}} = \begin{pmatrix} \vec{D} \\ 2\alpha\vec{B} \end{pmatrix} \tag{5}$$

From equation(2)-(4), it can obtained the following relationship

$$\vec{\mathcal{E}} = c\hat{k}\times(-i\sigma_2)\mathcal{M}\vec{\mathcal{E}} \tag{6}$$

Since $\vec{E}$ and $\vec{H}$ are not independent for a wave solution, we can express $\vec{\mathcal{E}}$ as

$$\vec{\mathcal{E}} = \begin{pmatrix} 2\alpha\vec{E} \\ c\varepsilon\hat{k}\times\vec{E} - \frac{2\alpha\theta}{2\pi}\vec{E} \end{pmatrix} \tag{7}$$

These results contain all the electric-magnetic relations of the TI in compact forms.

The Kerr and Faraday rotations at orthogonal incidence at the interface between two materials with different ε, μ and θ have been calculated in Ref.[5]. As a generalization, we now consider the corresponding problem in the case of oblique incidence, i.e. the angle of incidence is not restrict to zero.

Since Snell's law is unmodified even in the presence of a jump in the axion angle θ, the main task for us is to calculate the components of the reflected and transmitted electric fields. Consider light ray shown in Fig.[1], incident at an certain angle α' at a plane interface separating two such materials mentioned before, and linearly polarized in the y direction $\vec{E}_{in} = E_{in}\hat{y}$. The wave vector of incidence, reflection and refraction are expressed as

$$\vec{k}_{in} = k_{in}(\sin\alpha', 0, \cos\alpha'), \quad \vec{k}_r = k_r(\sin\alpha', 0, -\cos\alpha'), \quad \vec{k}_t = k_t(\sin\gamma', 0, \cos\gamma') \tag{8}$$

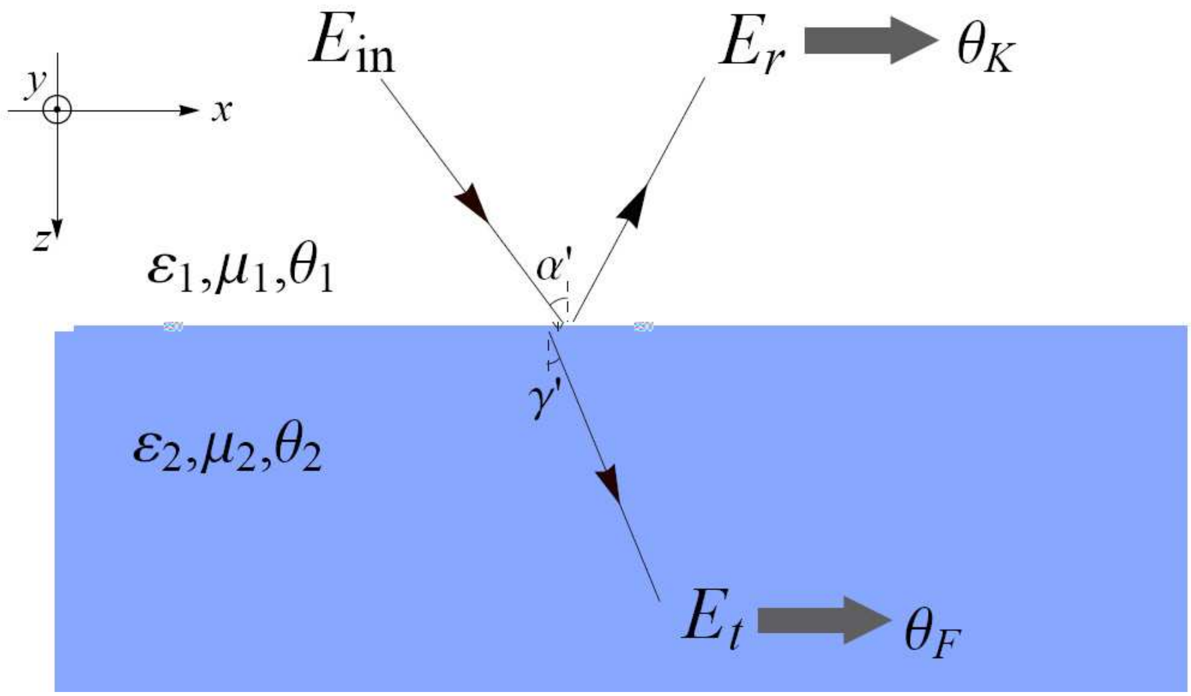

FIG. 1: Kerr and Faraday rotation at a single surface with oblique incidence

Because $\hat{k}_i \cdot \vec{\mathcal{E}}_i = 0$, the components of $\vec{\mathcal{E}}$ are not independent: $\mathcal{E}_r^x = \cot\alpha' \mathcal{E}_r^z$, $\mathcal{E}_t^x = -\cot\gamma' \mathcal{E}_t^z$. Then we can obtain

$$\tan\theta_K = \frac{\vec{E}_r \cdot (\hat{k}_r \times \hat{y})}{\vec{E}_r \cdot \hat{y}} = \frac{\cos\alpha' E_r^x + \sin\alpha' E_r^z}{E_r^y} = \frac{1}{\sin\alpha'}\frac{E_r^z}{E_r^y} \tag{9}$$

$$\tan\theta_F = \frac{\vec{E}_t \cdot (\hat{y} \times \hat{k}_t)}{\vec{E}_t \cdot \hat{y}} = \frac{\cos\gamma' E_t^x - \sin\gamma' E_t^z}{E_t^y} = -\frac{1}{\sin\gamma'}\frac{E_t^z}{E_t^y} \tag{10}$$

In this case, all the fields no longer parallel to the interface, so the boundary conditions are continuity of the parallel component of $\vec{\mathcal{E}} = \begin{pmatrix} 2\alpha\vec{E} \\ \vec{H} \end{pmatrix}$ as well as the vertical component of $\vec{\mathcal{D}} = \begin{pmatrix} \vec{D} \\ 2\alpha\vec{B} \end{pmatrix}$, which is different from and more complicated than the case of normal incidence. Two relations can be obtained

$$\hat{z} \times (\vec{\mathcal{E}}_{in} + \vec{\mathcal{E}}_r) = \hat{z} \times \vec{\mathcal{E}}_t \tag{11}$$
$$\hat{z} \cdot (\vec{\mathcal{D}}_{in} + \vec{\mathcal{D}}_r) = \hat{z} \cdot \vec{\mathcal{D}}_t \tag{12}$$

Using Eq.(6), we get the following equation,

$$\mathcal{T}_{12}[\hat{z} \times (\hat{k}_{in} \times \vec{\mathcal{E}}_{in} + \hat{k}_r \times \vec{\mathcal{E}}_r)] = \hat{z} \times (\hat{k}_t \times \vec{\mathcal{E}}_t) \tag{13}$$

where we define a transfer matrix as $\mathcal{T}_{ij} = \frac{c_i}{c_j}\mathcal{M}_j^{-1}\mathcal{M}_i = \frac{4\alpha^2}{Y_1 Y_2}\begin{pmatrix} \frac{Y_i^2}{4\alpha^2} - \frac{\theta_i(\theta_j-\theta_i)}{4\pi^2} & -\frac{\theta_j-\theta_i}{2\pi} \\ -\frac{\theta_j}{2\pi}\frac{Y_i^2}{4\alpha^2} + \frac{\theta_i}{2\pi}\frac{Y_j^2}{4\alpha^2} + \frac{\theta_i\theta_j(\theta_j-\theta_i)}{8\pi^3} & \frac{Y_j^2}{4\alpha^2} + \frac{\theta_j(\theta_j-\theta_i)}{4\pi^2} \end{pmatrix}$ and $Y_i = \sqrt{\frac{\varepsilon_i}{\mu_i}}$ denotes the admittance of material i.

From the constitutive relation, Eq.(12) is equivalent to

$$\frac{c_2}{c_1}\mathcal{T}_{12}\hat{z} \cdot (\vec{\mathcal{E}}_{in} + \vec{\mathcal{E}}_r^z) = \hat{z} \cdot \vec{\mathcal{E}}_t \tag{14}$$

We choose the independent equations of y- and z-components from Eqs.(11)-(14),

$$\frac{\cot\alpha'}{\cot\gamma'}(\mathcal{E}^z_{in}-\mathcal{E}^z_r)=\mathcal{E}^z_t \tag{15}$$

$$\mathcal{E}^y_{in}+\mathcal{E}^y_r=\mathcal{E}^y_t \tag{16}$$

$$\frac{c_2}{c_1}\mathcal{T}_{12}(\mathcal{E}^z_{in}+\mathcal{E}^z_r)=\mathcal{E}^z_t \tag{17}$$

$$\frac{\cos\alpha'}{\cos\gamma'}\mathcal{T}_{12}(\mathcal{E}^y_{in}-\mathcal{E}^y_r)=\mathcal{E}^y_t \tag{18}$$

where $\mathcal{E}^z_{in}=E\begin{pmatrix}0\\ Y_1\sin\alpha'\end{pmatrix}$ and $\mathcal{E}^y_{in}=2\alpha E\begin{pmatrix}1\\ -\frac{\theta_1}{2\pi}\end{pmatrix}$.

From above, we obtain the $\vec{\mathcal{E}}_r$ and $\vec{\mathcal{E}}_t$ simultaneously, here we give $\vec{\mathcal{E}}_r$ only,

$$\mathcal{E}^z_r=(1+\frac{\cos\gamma'}{\cos\alpha'}\mathcal{T}_{12})^{-1}(1-\frac{\cos\gamma'}{\cos\alpha'}\mathcal{T}_{12})\mathcal{E}^z_{in} \tag{19}$$

$$\mathcal{E}^y_r=-(1+\frac{\cos\alpha'}{\cos\gamma'}\mathcal{T}_{12})^{-1}(1-\frac{\cos\alpha'}{\cos\gamma'}\mathcal{T}_{12})\mathcal{E}^y_{in} \tag{20}$$

At last we get the polarization plane of reflected electric field rotated by an angle θ_K with

$$\tan\theta_K=\frac{4\alpha}{Y_2}\frac{\theta_2-\theta_1}{2\pi}\frac{\frac{\cos\alpha'}{\cos\gamma'}}{1+\frac{\cos\alpha'}{\cos\gamma'}(\frac{Y_2}{Y_1}-\frac{Y_1}{Y_2})-\frac{\cos^2\alpha'}{\cos^2\gamma'}}+o(\alpha^2) \tag{21}$$

In the limit of orthogonal incidence, i.e. $\alpha'\to 0$, we obtain $\tan\theta_K=4\alpha\frac{\theta_2-\theta_1}{2\pi}\frac{Y_1}{Y_2^2-Y_1^2}+o(\alpha^2)$, which is a direct consequence of a jump in θ, and is perfect agreement with the result in Ref. [5, 14].

3. KERR AND FARADAY ROTATION ON A TOPOLOGICAL INSULATOR THICK FILM AT OBLIQUE INCIDENCE

Now, as shown in Fig.[2], we consider a TI thick film of thickness L with optical constants $\varepsilon_2,\ \mu_2$ and axion angle θ deposited on a topological trivial insulating substrate with optical constant $\varepsilon_3,\ \mu_3$. The substrate is characterized by an axion angle $\theta_{sub}=2p\pi$, where p is an integer. Generally, only the change of the θ angle across an interface is physically important, which defines the Hall conductance of the interface[5]. In our setup, the two interfaces are defined by the difference of θ at the upper interface and $\theta_{sub}-\theta$ at the low interface. Therefore, p is a measure of the total Hall conductances of both surfaces. The vacuum outside the TI has $\varepsilon=\mu=1$ and trivial axion angle $\theta_{vac}=0$. Because the reflectivity at TI/vacuum interface is quite large, the effect of multiple reflections should be considered. For the oblique incidence, shown in Fig.[2], with the angle of incidence is α', we have

$$\vec{\mathcal{E}}(\vec{r},t)=\begin{cases}\vec{\mathcal{E}}_1^+e^{ik_1(x\sin\alpha'+z\cos\alpha')-i\omega t}+\vec{\mathcal{E}}_1^-e^{ik_1(x\sin\alpha'-z\cos\alpha')-i\omega t}, & z>0\\ \vec{\mathcal{E}}_2^+e^{ik_2(x\sin\beta'+z\cos\beta')-i\omega t}+\vec{\mathcal{E}}_2^-e^{ik_2(x\sin\beta'-z\cos\beta')-i\omega t}, & 0<z<L\\ \vec{\mathcal{E}}_3^+e^{ik_3(x\sin\gamma'+z\cos\gamma')-i\omega t}, & z>L\end{cases} \tag{22}$$

For the oblique incidence case, the boundary conditions boundary condition at $z=0$ and $z=L$ are the continuity of the parallel component of $\vec{\mathcal{E}}$ as well as the vertical component of $\vec{\mathcal{D}}$, using the fact that $k_i\sin\phi_i=\omega\frac{\sin\phi_i}{c_i}=const$, we get

$$\hat{z}\times(\vec{\mathcal{E}}_1^++\vec{E}_1^-)=\hat{z}\times(\vec{\mathcal{E}}_2^++\vec{\mathcal{E}}_2^-) \tag{23}$$

$$\hat{z}\times(\vec{\mathcal{E}}_2^+e^{ik_2L\cos\beta'}+\vec{\mathcal{E}}_2^-e^{-ik_2L\cos\beta'})=\hat{z}\times\vec{\mathcal{E}}_3^+e^{ik_3L\cos\gamma'} \tag{24}$$

$$\hat{z}\cdot(\vec{\mathcal{D}}_1^++\vec{\mathcal{D}}_1^-)=\hat{z}\cdot(\vec{\mathcal{D}}_2^++\vec{\mathcal{D}}_2^-) \tag{25}$$

$$\hat{z}\cdot(\vec{\mathcal{D}}_2^+e^{ik_2L\cos\beta'}+\vec{\mathcal{D}}_2^-e^{-ik_2L\cos\beta'})=\hat{z}\cdot\vec{\mathcal{D}}_3^+e^{ik_3L\cos\gamma'} \tag{26}$$

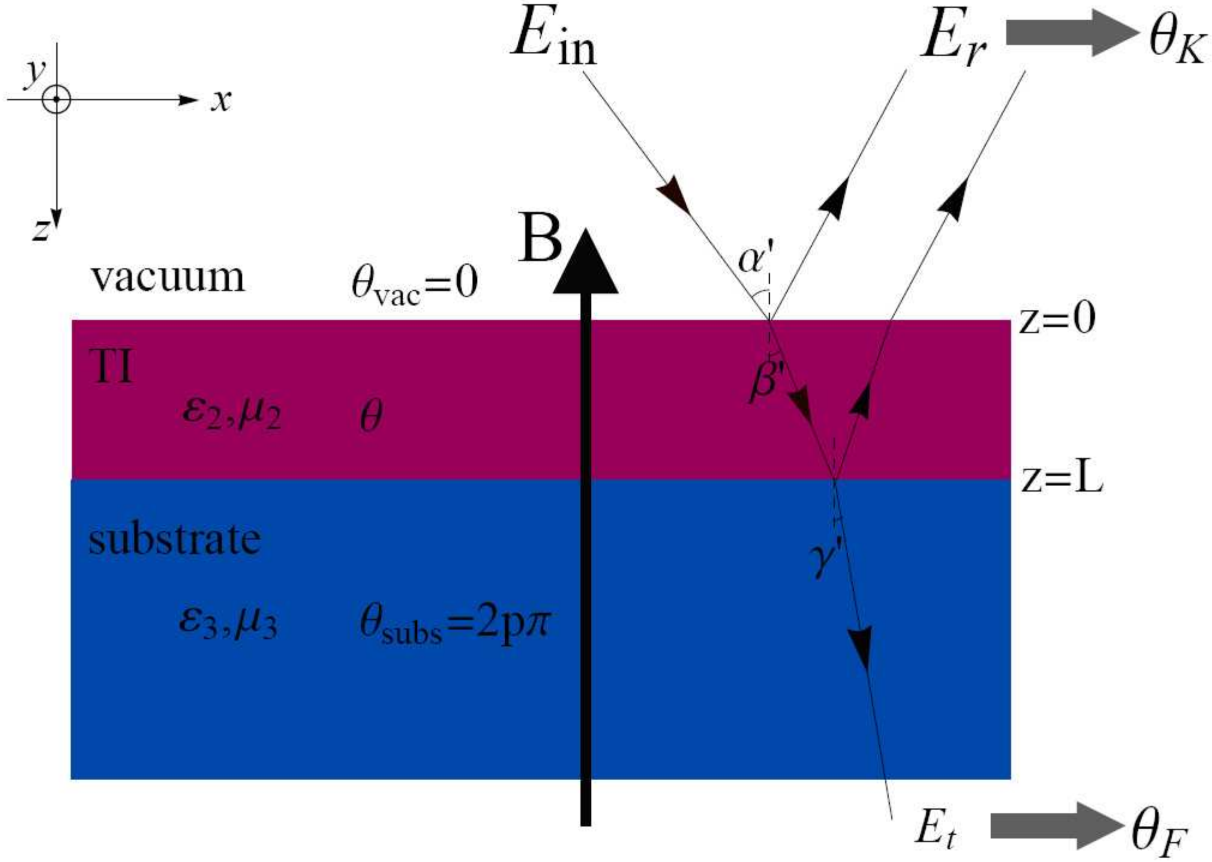

FIG. 2: Kerr and Karaday rotations at oblique incidence at a TI thick film

Similar to the method in section II, using Eq.(2) and (6), and choosing the independent z and y components of $\mathcal{E}$, we acquire two sets of equations.

For z component, we get

$$\frac{\cot\alpha'}{\cot\beta'}(\mathcal{E}_{1z}^{+}-\mathcal{E}_{1z}^{-})=\mathcal{E}_{2z}^{+}-\mathcal{E}_{2z}^{-} \tag{27}$$

$$\frac{c_2}{c_1}\mathcal{T}_{12}(\mathcal{E}_{1z}^{+}+\mathcal{E}_{1z}^{-})=\mathcal{E}_{2z}^{+}\mathcal{E}_{2z}^{-} \tag{28}$$

$$\frac{\cot\beta'}{\cot\gamma'}(\mathcal{E}_{2z}^{+}e^{ik_2L\cos\beta'}-\mathcal{E}_{2z}^{-}e^{-ik_2L\cos\beta'})=\mathcal{E}_{3z}^{+}e^{ik_3L\cos\gamma'} \tag{29}$$

$$\frac{c_3}{c_2}\mathcal{T}_{23}(\mathcal{E}_{2z}^{+}e^{ik_2L\cos\beta'}+\mathcal{E}_{2z}^{-}e^{-ik_2L\cos\beta'})=\mathcal{E}_{3z}^{+}e^{ik_3L\cos\gamma'} \tag{30}$$

For y component, we get

$$\mathcal{E}_{1y}^{+}+\mathcal{E}_{1y}^{-}=\mathcal{E}_{2y}^{+}+\mathcal{E}_{2y}^{-} \tag{31}$$

$$\frac{\cos\alpha'}{\cos\beta'}\mathcal{T}_{12}(\mathcal{E}_{1y}^{+}-\mathcal{E}_{1y}^{-})=\mathcal{E}_{2y}^{+}-\mathcal{E}_{2y}^{-} \tag{32}$$

$$\mathcal{E}_{2y}^{+}e^{ik_2L\cos\beta'}+\mathcal{E}_{2y}^{-}e^{-ik_2L\cos\beta'}=\mathcal{E}_{3y}^{+}e^{ik_3L\cos\gamma'} \tag{33}$$

$$\frac{\cos\beta'}{\cos\gamma'}\mathcal{E}_{2y}^{+}e^{ik_2L\cos\beta'}-\mathcal{E}_{2y}^{-}e^{-ik_2L\cos\beta'}=\mathcal{E}_{3y}^{+}e^{ik_3L\cos\gamma'} \tag{34}$$

After some algebra, we obtain

$$\mathcal{E}_{1z}^{-} = V_K \mathcal{E}_{1z}^{+}, \qquad e^{ik_3 L\cos\gamma'}\mathcal{E}_{3z}^{-} = U_F \mathcal{E}_{1z}^{+} \tag{35}$$

$$\mathcal{E}_{1y}^{-} = V_K' \mathcal{E}_{1y}^{+}, \qquad e^{ik_3 L\cos\gamma'}\mathcal{E}_{3y}^{+} = U_F' \mathcal{E}_{1z}^{+} \tag{36}$$

where

$$U_F = \frac{c_3}{c_2}[\mathbb{I} + \frac{\cos\gamma'}{\cos\beta'}\mathcal{T}_{23}\mathcal{Q}_{12}^{*}(\mathcal{P}_{12}^{*})^{-1}]^{-1}\mathcal{T}_{23}[\mathcal{Q}_{12} + \mathcal{Q}_{12}^{*}(\mathcal{P}_{12}^{*})^{-1}\mathcal{P}_{12}] \tag{37}$$

$$U_F' = \frac{\cos\beta'}{\cos\gamma'}[\mathbb{I} + \frac{\cos\beta'}{\cos\gamma'}\mathcal{N}_{12}(\mathcal{M}_{12}^{*})^{-1}]^{-1}\mathcal{T}_{23}[\mathcal{N}_{12} + \mathcal{N}_{12}^{*}(\mathcal{M}_{12}^{*})^{-1}\mathcal{M}_{12}] \tag{38}$$

$$U_F - \frac{\cot\beta'}{\cot\gamma'}\mathcal{P}_{12} = \frac{\cot\beta'}{\cot\gamma'}\mathcal{P}_{12}^{*}V_K \tag{39}$$

$$U_F' - \mathcal{M}_{12} = \mathcal{M}_{12}^{*}V_K' \tag{40}$$

where we define four complex matrices

$$\mathcal{P}_{12} = \frac{\cot\alpha'}{\cot\beta'}\cos(k_2 L\cos\beta')\mathbb{I} + i\frac{c_2}{c_1}\sin(k_2 L\cos\beta')\mathcal{T}_{12} \tag{41}$$

$$\mathcal{Q}_{12} = \frac{c_2}{c_1}\cos(k_2 L\cos\beta')\mathcal{T}_{12} + i\frac{\cot\alpha'}{\cot\beta'}\sin(k_2 L\cos\beta')\mathbb{I} \tag{42}$$

$$\mathcal{M}_{12} = \cos(k_2 L\cos\beta')\mathbb{I} + i\frac{\cos\alpha'}{\cos\beta'}\sin(k_2 L\cos\beta')\mathcal{T}_{12} \tag{43}$$

$$\mathcal{N}_{12} = \frac{\cos\alpha'}{\cos\beta'}\cos(k_2 L\cos\beta')\mathcal{T}_{12} + i\sin(k_2 L\cos\beta')\mathbb{I} \tag{44}$$

Then the Faraday and Kerr rotation can be given by the following

$$\tan\theta_F = -\frac{1}{\sin\gamma'}\frac{E_{3z}^{+}}{E_{3y}^{+}} = -\frac{Y_1}{2\alpha}\frac{\sin\alpha'}{\sin\gamma'}\frac{U_F^{12}}{U_F'^{11}} = -\frac{1}{2\alpha}\frac{\sin\alpha'}{\sin\gamma'}\frac{U_F^{12}}{U_F'^{11}} \tag{45}$$

$$\tan\theta_K = \frac{1}{\sin\alpha'}\frac{E_{1z}^{-}}{E_{1y}^{-}} = \frac{Y_1}{2\alpha}\frac{V_K^{12}}{V_K'^{11}} = \frac{1}{2\alpha}\frac{V_K^{12}}{V_K'^{11}} \tag{46}$$

In general, Eq.(45) and (46) depend on a complicated way on the optical constants of both the TI and the substrate, as well as on the TI film thickness L, the wave frequency ω and the angle of incidence. Since matrices $U_F(U_F')$ and $V_K(V_K')$ are complex, both Faraday and Kerr angle are complex, that means that the transmitted and reflected electric fields will acquire some ellipticity in addition to the rotation of the plane of polarization. But in the dc limit, i.e. $\omega \to 0$, all the values are reduced to real number, Eq.(45) and (46) become

$$\tan\theta_F = \frac{2\alpha p}{Y_3 + \frac{\cos\gamma'}{\cos\alpha'}} \tag{47}$$

$$\tan\theta_K = \frac{4\alpha p}{Y_3^2 - 1 + 4\alpha^2 p^2 + Y_3(\frac{\cos\gamma'}{\cos\alpha'} - \frac{\cos\alpha'}{\cos\gamma'})} \tag{48}$$

So in low frequency limit, both Faraday and Kerr rotations depend only on the optical constants of substrate, the angle of incidence, and the important parameter p. All the preview work Ref.[11, 12] are focus on the case of normal incidence, which can be directed acquired from Eqs.(47)-(48) by letting $\alpha' \to 0$.

4. GENERALIZED QUANTIZATION CONDITION IN TME

From Eqs.(37)-(46), it can be obviously seen that $\tan\theta_F, \tan\theta_K$ are periodic functions of the combination of $k_2 L\cos\beta'$ because of the periodicity of the matrices functions $\mathcal{P}_{12}, \mathcal{Q}_{12}, \mathcal{M}_{12}, \mathcal{N}_{12}$. More specifically, when $k_2 L\cos\beta' = n\pi, (n \in \mathbb{Z})$, both $\tan\theta_F$, $\tan\theta_K$ are equal to the results in dc limit separately. The condition $k_2 L\cos\beta' = n\pi$ is equivalently to the constraint to the thickness of TI film, i.e.

$$L\cos\beta' = n\frac{\lambda_2}{2}, \qquad (n \in \mathbb{Z}) \tag{49}$$

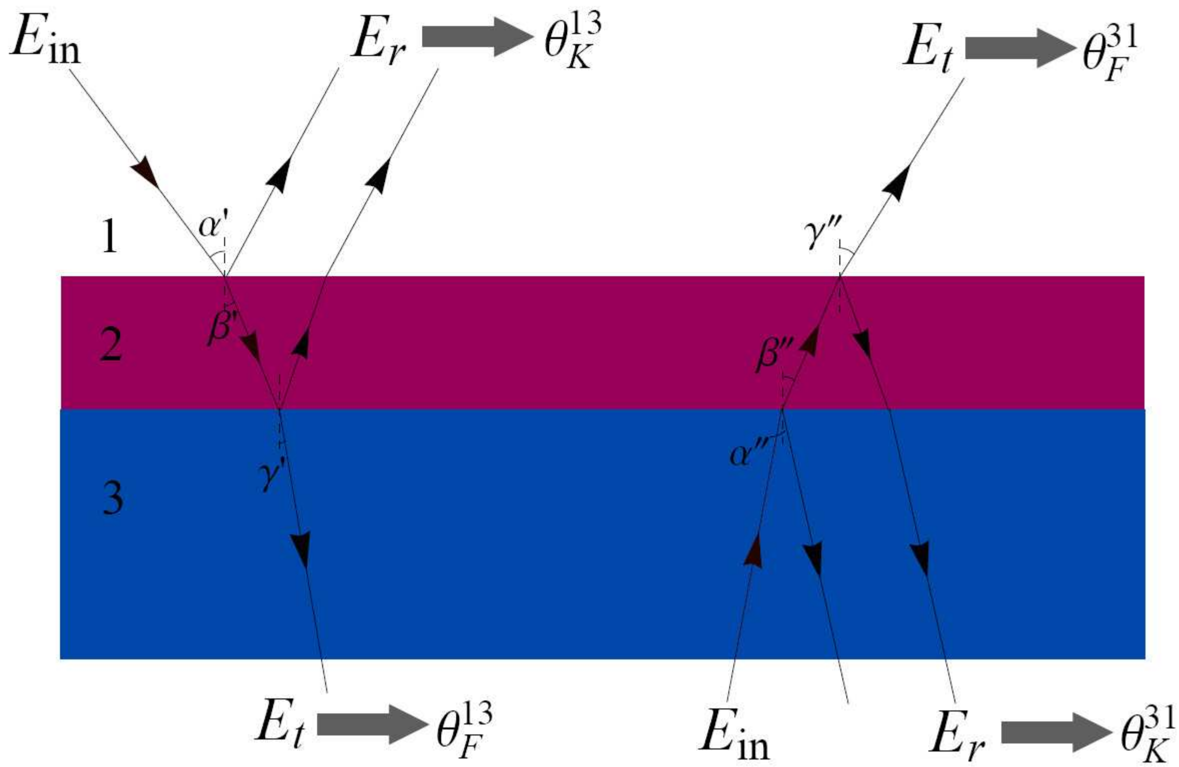

FIG. 3: Kerr and Faraday measurements in both directions

where $\lambda_2 = \frac{2\pi c}{\omega\sqrt{\varepsilon_2\mu_2}}$ is the optical wavelength in the TI. On the other hand, these certain values are exactly corresponding to the minima of reflectivity. In Ref.[11], the author proposed an experiment which need to tune the photon frequency ω or the thickness of TI film L to that special values in order to observing that minima of reflectivity, but neither ω nor L can be continuously tuned by convenient ways. But in the case of oblique incidence, we introduce an extra parameter, α', which can be continuously changed conveniently in experiments. In view of this reason, we propose a new experiment scenario: firstly, one choose a TI film with appropriate thickness and depose it on a topological trivial substrate, and then tune the angle of incidence from 0 to $\frac{\pi}{2}$ smoothly, measure θ_F and θ_K when it occurs at reflectivity minima. At these values, both $\tan\theta_F$ and $\tan\theta_K$ have the same expression with Eq.(47) and (48), and

$$\tan\theta_F = \frac{2\alpha p}{Y_3 + \frac{\cos\gamma'}{\cos\alpha'}}, \quad \tan\theta_K = \frac{4\alpha p}{Y_3^2 - 1 + 4\alpha^2p^2 + Y_3(\frac{\cos\gamma'}{\cos\alpha'} - \frac{\cos\alpha'}{\cos\gamma'})}, \quad L\cos\beta' = n\frac{\lambda_2}{2} \tag{50}$$

which are also dependent on the angle of incidence. For example, if the thickness of TI film is 3.2 (with the unit of $\frac{\lambda_2}{2}$), then we can strictly get reflectivity minima three times altogether in the process of tuning angle of incidence.

In the case of orthogonal incidence, one can eliminate the explicit dependence on the substrate property Y_3 by combining θ_F and θ_K and then obtain a topological quantization condition in TME, but unfortunately, it does't work in the case of oblique incidence for the emergence of an extra parameter α'. However, this difficulty can be solved if we measure the Faraday and Kerr angles in both direction. We will elaborate this idea in the following discussion. We denote by θ_F^{13} and θ_K^{13} the Faraday and Kerr angles defined previously in Eq.(47) and (48), respectively. While θ_F^{31} and θ_K^{31} represent the Faraday and Kerr angles for light ray travelling in the opposite direction shown in Fig.[3]. When light incident from the substrate and get reflectivity minima, we obtain

$$\tan\theta_F^{31} = \frac{2\alpha p}{1 + \frac{\cos\gamma''}{\cos\alpha''}(Y_3 + \frac{4\alpha^2p^2}{Y_3})} \tag{51}$$

$$\tan\theta_K^{31} = -\frac{4\alpha p Y_3}{Y_3^2 - 1 + 4\alpha^2p^2 + Y_3(\frac{\cos\alpha''}{\cos\gamma''} - \frac{\cos\gamma''}{\cos\alpha''})} \tag{52}$$

where denote by $\alpha'', \beta'', \gamma''$ the angle in substrate, TI film and vacuum, separately. Eq.(51) and (52) independent of TI properties but depend on the angle of incidence. The condition of reflection minima is $k_2 L \cos\beta'' = m\pi$, i.e.

$$L\cos\beta'' = m\frac{\lambda_2}{2}, \quad (m \in \mathbb{Z}) \tag{53}$$

Comparing Eqs.(49) and Eqs.(53), we can find that the sets of angles, $\alpha'(\beta', \gamma')$ and $\alpha''(\beta'', \gamma'')$ exist a one to one corresponding relation, i.e.

$$\beta'' = \beta', \quad \alpha'' = \gamma', \quad \gamma'' = \alpha', \quad (m = n) \tag{54}$$

substitute Eq.(54) into Eq.(51) and (52), we find

$$\tan\theta_F^{31} = \frac{2\alpha p}{1 + \frac{\cos\alpha'}{\cos\gamma'}(Y_3 + \frac{4\alpha^2 p^2}{Y_3})} \tag{55}$$

$$\tan\theta_K^{31} = -\frac{4\alpha p Y_3}{Y_3^2 - 1 + 4\alpha^2 p^2 + Y_3(\frac{\cos\gamma'}{\cos\alpha'} - \frac{\cos\alpha'}{\cos\gamma'})} \tag{56}$$

comparing Eq.(48) and Eq.(56), we obtain that Y_3 can be given as

$$Y_3 = -\cot\theta_K^{13}\tan\theta_K^{31} \tag{57}$$

and this relation is also true in the case of orthogonal incidence. Moreover, the four angles, i.e. θ_K^{13}, θ_F^{13}, θ_K^{31}, θ_F^{31}, can be combined to obtain a universal result independent of both TI and substrate properties,

$$\frac{\cot\theta_F^{13}\tan\theta_F^{31} - \cot\theta_K^{13}\tan\theta_K^{31}}{\cot\theta_F^{13} + \tan\theta_K^{13}\tan\theta_F^{31}\cot\theta_K^{31}} = 2\alpha p, \quad p \in \mathbb{Z} \tag{58}$$

However, it should be pointed out that Eq.(58) does not depend explicitly on the angle of incidence, but the two sets of rotation angles in both direction must be measured at certain condition. Specifically, if θ_F^{13} and θ_K^{13} are measured at the nth time of reflectivity minima when the angle of incidence α' continuously tuned from 0 to $\frac{\pi}{2}$, then θ_F^{31} and θ_K^{31} must be measured at the nth time of reflectivity minima also, otherwise, $\alpha'' = \gamma'$. Equation (58) provides a universal topological quantization of the TME in units of the fine structure constant α, independent of material properties such as ε and μ, and it applies to both normal incidence and oblique incidence, which is the central result of our work.

Similar to the case of normal incidence, at reflectivity minima our generalized quantization condition depends only on p and not θ. Therefore, this experiment measures the total Hall conductance of both interfaces. The problem measuring θ, or the Hall conductances of each surfaces separately, can be solved by the same method of Ref.[11]. The basic idea is: to obtain the axion angle θ we can do another optical measurement performed at reflectivity maxima, $L\cos\beta' = (n + \frac{1}{2})\frac{\lambda_2}{2}, \;\; n \in \mathbb{Z}$, and the rotation angle corresponding to reflectivity maxima depend explicity on the TI axion angle θ. And then using all the measurements to construct a universal function of a single variable $f(\theta)$ which crosses zero at the value of the bulk axion angle θ with no 2π ambiguity. The zero crossing point is independent of material parameters and, together with Eq.(58), provides a universal experimental demonstration of the universal quantization of the TME in the bulk of a TI.

5. CONCLUSIONS

In this article, we work out manifestly $SL(2, \mathbb{Z})$ electric-magnetic duality covariant expression for the Kerr and Faraday angles at oblique incidence at a single surface between a trivial insulator and a semi-infinite topological insulator, as well as at a topological insulator thick film with two surfaces. When light incident at a topological insulator thick film with a finite incidence angle, we give a generalized topological quantization condition by combining two sets of Faraday and Kerr angles in both direction which are all measured at reflectivity minima. The generalized topological quantization condition obtained here can be easier to realize experimentally compared with an earlier proposal[11], since the incidence angle can be continuously tuned.

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